PRINCIPLES OF ANALYSIS
LECTURE 22 - INTEGRATION

PAUL L. BAILEY

Let $a, b \in \mathbb{R}$ with $a < b$. A *partition* of $[a, b]$ is a finite set $\{t_0, t_1, \dots, t_n\}$ with $a = t_0 < t_1 < \cdots < t_n = b$.

Let $\mathcal{N}(a, b)$ denote the set of all partitions of $[a, b]$. Note that this set is partially ordered by inclusion. If $P, Q \in \mathcal{N}(a, b)$ and $P \subset Q$, we say that Q is a *refinement* of P.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. Set

$$M_f(P, i) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \quad \text{and} \quad m_f(P, i) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

The *upper Rieman sum* of f with respect P is

$$U_f(P) = \sum_{i=1}^{n} M_f(P, i)(x_i - x_{i-1}),$$

and the *lower Rieman sum* of f with respect to P is

$$L_f(P) = \sum_{i=1}^{n} m_f(P, i)(x_i - x_{i-1}).$$

Since f is bounded, there exist $m, M \in \mathbb{R}$ with $m < M$ such that $f(x) \in [m, M]$ for every $x \in [a, b]$. Thus

$$m(b - a) \leq L_f(P) \leq U_f(P) \leq M(b - a).$$

Moreover, if Q is a refinement of P, then

$$L_f(P) \leq L_f(Q) \leq U_f(Q) \leq U_f(P).$$

The *upper Riemann integral* of f is

$$\overline{\int_a^b} f\,dx = \inf\{U_f(P) \mid P \in \mathcal{N}(a, b)\},$$

and the *lower Riemann integral* of f is

$$\underline{\int_a^b} f\,dx = \sup\{L_f(P) \mid P \in \mathcal{N}(a, b)\}.$$

Note that the upper and lower Riemann integrals exist for any bounded function; in fact, if $f(x) \in [m, M]$ for every $x \in [a, b]$, then

$$m(b - a) \leq \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx \leq M(b - a).$$

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We say that f is *Riemann integrable* on $[a,b]$ if $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$. The common value is called the *Riemann integral*, and is denotedby $\int_a^b f\,dx$.

The adjective *Riemann* precedes the word *integrable* because there are other sorts of integrals which are of as much or more importance to theoretical mathematics as the Riemann integral. Predominant among these is the *Lebesque* integral, which is defined by splitting up the range of the function f instead of its domain. However, we will not study Lebesque integrable functions, and the modifier Riemann becomes superfluous for us. Thus we will call a Riemann integrable function simply *integrable*.

Proposition 1. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then the following conditions are equivalent:*

- **(a)** *f is integrable on $[a,b]$;*
- **(b)** $\overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx = 0$*;*
- **(c)** $\inf\{U_f(P) - L_f(P) \mid P \in \mathcal{N}(a,b)\} = 0$*;*
- **(d)** $\forall \epsilon > 0\, \exists P \in \mathcal{N}(a,b) \ni U_f(P) - L_f(P) < \epsilon$*.*

Proof. It is obvious that **(a)** is equivalent to **(b)**. Also, that **(c)** implies **(d)** is clear. That **(d)** implies **(c)** follows immediately from the fact that $U_f(P) \geq L_f(P)$ for every $P \in \mathcal{N}(a,b)$.

Suppose that f is integrable on $[a,b]$, and set $I = \int_a^b f\,dx$, that is,

$$\sup\{L_f(P) \mid P \in \mathcal{N}(a,b)\} = I = \inf\{U_f(P) \mid P \in \mathcal{N}(a,b)\}.$$

Let $\epsilon > 0$. Then there exist partitions P_1 and P_2 of $[a,b]$ such that

$$U_f(P_1) - \frac{\epsilon}{2} < I < L_f(P_2) + \frac{\epsilon}{2}.$$

Let $P = P_1 \cup P_2$; then P is a common refinement of P_1 and P_2, and

$$U_f(P) - \frac{\epsilon}{2} \leq U_f(P_1) - \frac{\epsilon}{2} < I < L_f(P_2) + \frac{\epsilon}{2} \leq L_f(P) + \frac{\epsilon}{2}.$$

This implies that

$$U_f(P) - L_f(P) < \epsilon,$$

which shows that **(a)** implies **(d)**.

Now suppose that condition **(d)** holds. Let $\epsilon > 0$, and let $P \in \mathcal{N}(a,b)$ such that $U_f(P) - L_f(P) < \epsilon$. Now $\overline{\int_a^b} f\,dx \leq U_f(P)$, and $\underline{\int_a^b} f\,dx \geq L_f(P)$. Subtracting these inequalities yields

$$0 \leq \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \leq U_f(P) - L_f(P) < \epsilon.$$

Since ϵ is arbitrary, this proves **(b)**. □

Example 1. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x$. Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[0, 1]$. Then

$$U_f(P) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}) \quad \text{and} \quad L_f(P) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1})$$

Then

$$U_f(P) - L_f(P) = \sum_{i=1}^{n} [x_i(x_i - x_{i-1}) - x_{i-1}(x_i - x_{i-1})] = \sum_{i=1}^{n} (x_i - x_{i-1})^2.$$

Let $\epsilon > 0$ and let n be so large that $n > \frac{1}{\epsilon}$. Define a partition P by $P = \{\frac{k}{n} \mid k = 0, \ldots, n\}$. Then

$$U_f(P) - L_f(P) = \sum_{i=1}^{n} \frac{1}{n^2} = \frac{1}{n} < \epsilon.$$

By the previous proposition, f is integrable.

Example 2. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[0, 1]$. Then for every i, $M_f(P, i) = 1$ and $m_f(P, i) = 0$, so $U_f(P) = 1$ and $L_f(P) = 0$. Therefore $\overline{\int_0^1} f\, dx = 1$ and $\underline{\int_0^1} f\, dx = 0$, so f is not integrable.

Example 3. Define $q : \mathbb{Q} \to \mathbb{Z}^+$ by

$$q(x) = \min\{b \in \mathbb{Z}^+ \mid x = \frac{a}{b} \text{ for some } a \in \mathbb{Z}\}.$$

Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{q(x)} & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Since every interval contains an irrational number, it is clear that $\underline{\int}_0^1 f\, dx = 0$.

Therefore, if f is integrable, we would have $\int_a^b f\, dx = 0$. We wish to show that the upper Riemann integral is zero.

Let $\epsilon > 0$. We construct a partition P of $[0, 1]$ such that $U_f(P) < \epsilon$.

There are only finitely many rational numbers $t \in (0, 1)$ such that $\frac{1}{q(t)} \geq \frac{\epsilon}{2}$; let $\{t_1, \dots, t_m\}$ be the set of such numbers, with $t_i < t_{i+1}$. Set

$$h = (\min\{t_{i+1} - t_i\} \cup \{\frac{\epsilon}{m}, 1 - t_m\})/2.$$

Then the intervals of the form $[t_i, t_i + h]$ are disjoint. Set $x_0 = 0$ and $x_{2m+1} = 1$, and for $i = 1, \dots, m$, set $x_{2i-1} = t_i$ and $x_{2i} = t_i + h$.

Set $n = 2m + 1$. Now $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[0, 1]$. For i odd, then $M_f(P, i) < \frac{\epsilon}{2}$. For i even, then $(x_i - x_{i-1}) \leq \frac{\epsilon}{2m}$. Thus

$$\begin{aligned} U_f(P) &= \sum_{i=1}^{n} M_f(P, i)(x_i - x_{i-1}) \\ &= \sum_{\text{odd}} M_f(P, i)(x_i - x_{i-1}) + \sum_{\text{even}} M_f(P, i)(x_i - x_{i-1}) \\ &< \frac{\epsilon}{2} \sum_{\text{odd}} (x_i - x_{i-1}) + h \sum_{\text{even}} M_f(P, i) \\ &< \frac{\epsilon}{2} + hm \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Let $f : [a, b] \to \mathbb{R}$. We say that f is *increasing* on $[a, b]$ if for every $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Proposition 2. *Let $f : [a, b] \to \mathbb{R}$ be increasing. Then f is integrable on $[a, b]$.*

Proof. Since f is increasing, $f(x) \in [f(a), f(b)]$ for every $x \in [a, b]$. In particular, f is bounded. Set $B = f(b) - f(a)$.

Let $P = \{x_0, \dots, x_n\}$ be any partition of $[a, b]$. Since f is increasing, we have $M_f(p, i) = \max\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(x_i)$, and $m_f(P, I) = \min\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(x_{i-1})$. Then

$$U_f(P) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \quad \text{and} \quad L_f(P) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}),$$

so $U_f(P) - L_f(P) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))(x_i - x_{i-1})$.

Let $\epsilon > 0$, and let $k = \frac{\epsilon}{2B}$ so that $0 < kB < \epsilon$. Choose a partition $P = \{x_0, x_1, \dots, x_n\}$ such that $x_i - x_{i-1} < k$. Then

$$\begin{aligned} U_f(P) - L_f(P) &\leq \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))k \\ &= k\sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) \\ &= kM < \epsilon. \end{aligned}$$

Thus f is integrable. □

Proposition 3. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable on $[a, b]$.*

Proof. Let $\epsilon > 0$; we wish to find a partition P such that $U_f(P) - L_f(P) < \epsilon$).

Since f is continuous and $[a, b]$ is compact, the image is also compact, and in particular, f is bounded on $[a, b]$. Moreover, f is uniformly continuous on $[a, b]$, so there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$.

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a, b]$ such that $|x_i - x_{i-1}| < \delta$. There exist $s_i, t_i \in [x_{i-1}, x_i]$ such that $f(s_i) = m_f(P, i)$ and $f(t_i) = M_f(P, i)$. Since $|s_i - t_i| < \delta$, we have $|f(t_i) - f(s_i)| < \epsilon/(b - a)$. Thus

$$\begin{aligned} U_f(P) - L_f(P) &= \sum_{i=1}^{n}(f(t_i) - f(s_i))(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} \frac{\epsilon}{b-a}(x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a}\sum_{i=1}^{n}(x_i - x_{i-1}) \\ &= \epsilon. \end{aligned}$$

Thus f is integrable. □

Department of Mathematics and CSCI, Southern Arkansas University
E-mail address: `plbailey@saumag.edu`